Lecture 4: September 9

Families of varieties and Hodge structures. Our topic today is variations of Hodge structure. Suppose we have a family of n-dimensional compact complex manifolds parametrized by a curve; as in algebraic geometry, this means concretely that we have a holomorphic mapping

$$f \colon X \to B$$

from a complex manifold X of dimension $n+1$ to a complex manifold B of dimension 1. We assume that f is proper and submersive, so that all the fibers $X_b = f^{-1}(b)$ are compact complex manifolds of dimension n. How do the cohomology groups of the fibers depend on $b \in B$? Also, assuming that all the fibers are Kähler manifolds, how do their Hodge structures vary? This question was studied systematically by Phillip Griffiths in the late 1960s.

Let me briefly summarize the main results, from a "topological" point of view; afterwards, we will reformulate and prove everything in a more algebraic way. First, one has Ehresmann's lemma, which says that all the fibers X_b are diffeomorphic to each other. More precisely, if $U \subseteq X$ is a neighborhood of some point $b \in B$ isomorphic to a disk, then $f^{-1}(U)$ is isomorphic, as a smooth manifold, to the product $U \times X_b$. This implies that the cohomology groups $H^k(X_b, \mathbb{C})$ are locally constant on B. In sheaf theory notation, the k-th higher direct image sheaf $R^k f_* \mathbb{C}$ is locally constant, with stalks

$$(R^k f_* \mathbb{C})_b \cong H^k(X_b, \mathbb{C}).$$

Now let us suppose that X is a Kähler manifold, with Kähler form ω; each fiber X_b is then also a Kähler manifold, with Kähler form $\omega_b = \omega\big|_{X_b}$. As we have seen,

$$H^k(X_b, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X_b)$$

has a polarized Hodge structure of weight k. Examples show that the Hodge decomposition is typically *not* locally constant on B: in many interesting cases, such as curves or K3-surfaces, the Hodge decomposition determines the complex structure up to isomorphism; if the family is not locally trivial, the Hodge decomposition therefore has to vary in a nontrivial way. To study this variation, we need to go from the locally constant sheaf $R^k f_* \mathbb{C}$ to the associated holomorphic vector bundle

$$\mathscr{V}^k = \mathscr{O}_B \otimes_{\mathbb{C}} R^k f_* \mathbb{C}.$$

To be precise, $\mathscr{V}^k$ is the locally constant sheaf associated to the holomorphic vector bundle with fibers $H^k(X_b, \mathbb{C})$; this vector bundle has the same (locally constant) transition functions as $R^k f_* \mathbb{C}$. Because $\mathscr{V}^k$ comes from a locally constant sheaf, it has a natural connection

$$\nabla \colon \mathscr{V}^k \to \Omega^1_B \otimes_{\mathscr{O}_B} \mathscr{V}^k,$$

called the *Gauss-Manin connection.* Namely, any local section $s \in \mathscr{V}^k$ can be written in the form $s = \sum g_i \otimes \sigma_i$ for holomorphic functions $g_i \in \mathscr{O}_B$ and sections $\sigma_i \in R^k f_* \mathbb{C}$, and then

$$\nabla(s) = \sum dg_i \otimes \sigma_i.$$

To say that ∇ is a connection means that it satisfies the Leibniz rule

$$\nabla(gs) = dg \otimes s + g\nabla(s)$$

for local sections $g \in \mathscr{O}_B$ and $s \in \mathscr{V}^k$. The Gauss-Manin connection is always a flat connection – but since $\dim B = 1$, this condition is vacuous in our setting. We can recover the locally constant sheaf

$$R^k f_* \mathbb{C} \subseteq \mathscr{V}^k$$

as the subsheaf of ∇-flat sections.

Consider again the Hodge decomposition

$$H^k(X_b, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X_b).$$

One can show that the Hodge numbers $\dim H^{p,q}(X_b)$ are constant in $b \in B$: by general theory, each function $b \mapsto \dim H^{p,q}(X_b)$ is upper semicontinuous, and their sum equals the constant function $b \mapsto \dim H^k(X_b, \mathbb{C})$. However, the Hodge decomposition does *not* vary holomorphically with $b \in B$, meaning that the subspaces $H^{p,q}(X_b)$ do not form holomorphic subbundles of $\mathscr{V}^k$. The reason is easy to see: the definition of $H^{p,q}(X_b)$ involves forms of type (p,q), and because of the anti-holomorphic differentials, this is clearly not a holomorphic condition. The solution is to consider instead the *Hodge filtration*

$$F^p H^k(X_b, \mathbb{C}) = \bigoplus_{i \geq p} H^{i,k-i}(X_b).$$

The Hodge filtration is very natural from several points of view: for example, the E_1-degeneration of the Hodge-de Rham spectral sequence

$$E_1^{p,q} = H^q(X_b, \Omega^p_{X_b}) \Longrightarrow H^{p+q}(X_b, \mathbb{C})$$

says exactly that the Hodge filtration is the filtration induced by the spectral sequence. Without extra information (such as a real structure or a polarization), one cannot recover the Hodge decomposition from the Hodge filtration, but let us defer this question until later. Anyway, Griffiths proved that the vector spaces $F^p H^k(X_b, \mathbb{C})$ fit together into holomorphic subbundles

$$F^p \mathscr{V}^k \subseteq \mathscr{V}^k,$$

called the *Hodge bundles.* He also found the following famous relation:

$$\nabla(F^p \mathscr{V}^k) \subseteq \Omega^1_B \otimes_{\mathscr{O}_B} F^{p-1} \mathscr{V}^k$$

In other words, applying the Gauss-Manin connection to a section of $F^p \mathscr{V}^k$ does not produce an arbitrary section of $\Omega^1_B \otimes \mathscr{V}^k$, but it can only move you up to the next Hodge bundle. Griffiths called this fact the "infinitesimal period relation", but it nowadays known as *Griffiths transversality.*

Lastly, how about the polarizations? Since the Kähler forms ω_b are restrictions of a Kähler form on X, the Lefschetz operators $L_b \colon H^k(X_b, \mathbb{C}) \to H^{k+2}(X_b, \mathbb{C})$ are locally constant, and therefore define morphisms

$$L \colon R^k f_* \mathbb{C} \to R^{k+2} f_* \mathbb{C}.$$

Similarly, integration over the fibers defines a hermitian pairing

$$R^{n+k} f_* \mathbb{C} \otimes_{\mathbb{C}} \overline{R^{n-k} f_* \mathbb{C}} \to \mathbb{C},$$

where the bar over a sheaf of complex vector spaces means that we replace every complex vector space by its conjugate. As explained in Lecture 3, we can put these two facts together and obtain a hermitian pairing

$$S^k \colon R^k f_* \mathbb{C} \otimes_{\mathbb{C}} \overline{R^k f_* \mathbb{C}} \to \mathbb{C}$$

that polarizes the Hodge structures on the stalks $(R_k f_* \mathbb{C})_b \cong H^k(X_b, \mathbb{C})$. Since I prefer to work with the holomorphic vector bundle $\mathscr{V}^k$, let me consider instead the induced hermitian pairing

$$S^k \colon \mathscr{V}^k \otimes_{\mathbb{C}} \overline{\mathscr{V}^k} \to \mathscr{C}^\infty_B$$

with values in the sheaf of C^∞-functions on B. The formula is

$$S^k\left(\sum g_i'\otimes\sigma_i', \sum g_j''\otimes\sigma_j''\right) = \sum g_i'\overline{g_j''}S^k(\sigma_i',\sigma_j''),$$

for local sections $\sigma_i',\sigma_j''\in R^kf_*\mathbb{C}$ and $g_i',g_j''\in\mathscr{O}_B$. By construction, this pairing is *flat*, meaning that if $s',s''\in\mathscr{V}^k$ are local sections, one has

$$d\,S^k(s',s'') = S^k(\nabla s',s'') + S^k(s',\nabla s''),$$

where the d on the left-hand side is the exterior derivative.

Algebraic construction of the flat bundle. All the above results are based on the topological fact that the fibers X_b are diffeomorphic to each other. There is another approach, due to Nick Katz and Tadao Oda, that uses algebraic methods. The idea is to construct the holomorphic vector bundle $\mathscr{V}^k$ and the Gauss-Manin connection directly, without going through the locally constant sheaf $R^kf_*\mathbb{C}$. This has the advantage of showing very clearly where Griffiths transversality comes from.

As before, let $f\colon X\to B$ be a holomorphic mapping from a complex manifold X of dimension $n+1$ to a complex manifold B of dimension 1, and suppose that f is proper and submersive. By the holomorphic Poincaré lemma, the complex

$$0\to\mathbb{C}\to\mathscr{O}_X\to\Omega_X^1\to\cdots\to\Omega_X^{n+1}\to 0$$

is exact, and so the holomorphic de Rham complex $\Omega_X^\bullet$ is a resolution of the constant sheaf $\mathbb{C}$. Using hypercohomology, this means that

$$R^kf_*\mathbb{C}\cong R^kf_*\Omega_X^\bullet.$$

The Katz-Oda construction is based on the relative version of the de Rham complex. Since f is submersive, we have a short exact sequence

$$0\to f^*\Omega_B^1\to\Omega_X^1\to\Omega_{X/B}^1\to 0,$$

with $\Omega_{X/B}^1$ locally free of rank n. Taking wedge powers, and remembering that Ω_B^1 is a line bundle, we get for each p a short exact sequence

$$0\to f^*\Omega_B^1\otimes_{\mathscr{O}_X}\Omega_{X/B}^{p-1}\to\Omega_X^p\to\Omega_{X/B}^p\to 0,$$

These fit together into a short exact sequence of complexes

$$0\to f^*\Omega_B^1\otimes_{\mathscr{O}_X}\Omega_{X/B}^{\bullet-1}\to\Omega_X^\bullet\to\Omega_{X/B}^\bullet\to 0. \tag{4.1}$$

By the relative version of the holomorphic Poincaré lemma, the complex

$$0\to f^{-1}\mathscr{O}_B\to\mathscr{O}_X\to\Omega_{X/B}^1\to\cdots\to\Omega_{X/B}^n\to 0$$

is exact, and so the relative de Rham complex resolves the inverse image sheaf $f^{-1}\mathscr{O}_B$ (whose sections are those holomorphic functions on X that are constant along the fibers of f). The proof uses the fact that, in local coordinates, f is represented by a coordinate projection $\mathbb{C}^{n+1}\to\mathbb{C}$, but it does not need any global facts about the topology of X. Anyway, we obtain

$$R^kf_*\Omega_{X/B}^\bullet\cong R^kf_*\big(f^{-1}\mathscr{O}_B\big)\cong\mathscr{O}_B\otimes_{\mathbb{C}}R^kf_*\mathbb{C},$$

by a version of the projection formula.

Now let us turn things around and define

$$\mathscr{V}^k = R^kf_*\Omega_{X/B}^\bullet.$$

What can we say about $\mathscr{V}^k$? By a famous theorem due to Hans Grauert, the higher direct image of a coherent sheaf under a proper holomorphic mapping is again coherent. This result extends without problems to complexes of coherent sheaves (using a spectral sequence argument), and so each $\mathscr{V}^k$ is a coherent sheaf of $\mathscr{O}_B$-modules. It is also not hard to construct a connection on $\mathscr{V}^k$. Indeed, consider the long exact sequence for the higher direct image sheaves of (4.1):

$$\cdots \to R^k f_* \Omega_X^\bullet \to R^k f_* \Omega_{X/B}^\bullet \to R^{k+1}\big(f^*\Omega_B^1 \otimes_{\mathscr{O}_X} \Omega_{X/B}^{\bullet-1}\big) \to \cdots$$

After using the projection formula and the isomorphisms from above, this becomes

$$\cdots \to R^k f_* \mathbb{C} \to \mathscr{V}^k \to \Omega_B^1 \otimes_{\mathscr{O}_B} \mathscr{V}^k \to \cdots \tag{4.2}$$

and so the connecting morphism gives us a $\mathbb{C}$-linear morphism

$$\nabla \colon \mathscr{V}^k \to \Omega_B^1 \otimes_{\mathscr{O}_B} \mathscr{V}^k$$

Note that ∇ is not $\mathscr{O}_B$-linear, because the differentials in the de Rham complex $\Omega_X^\bullet$ are not $\mathscr{O}_B$-linear. I will leave it as an exercise to check that ∇ is a connection.

Exercise 4.1. Verify that ∇ satisfies the Leibniz rule

$$\nabla(gs) = dg \otimes s + g\nabla(s)$$

for local sections $g \in \mathscr{O}_B$ and $s \in \mathscr{V}^k$.

We can use the existence of the connection to show that $\mathscr{V}^k$ is actually locally free. This is based on the following lemma from $\mathscr{D}$-module theory. We only need the 1-dimensional version here, but the general statement is that, on any complex manifold B, a coherent $\mathscr{O}_B$-module with a flat connection must be locally free. Some of you may remember this result from my course on $\mathscr{D}$-modules last semester.

Lemma 4.3. *Let B be a complex manifold of dimension 1, and $\mathscr{F}$ a coherent sheaf of $\mathscr{O}_B$-modules. If $\mathscr{F}$ admits a connection*

$$\nabla \colon \mathscr{F} \to \Omega_B^1 \otimes_{\mathscr{O}_B} \mathscr{F}$$

then $\mathscr{F}$ is locally free.

Proof. I will give a simplified proof that only works in dimension 1, but that shows the underlying idea very clearly. The problem is local, and so we let $R = \mathscr{O}_{B,b}$ be the local ring at a point $b \in B$; of course, R is isomorphic to the ring of convergent power series in t, where t is a local coordinate at b. By assumption, the stalk $M = \mathscr{F}_b$ is a finitely generated R-module, and if we apply the connection to the vector field $\partial_t = \partial/\partial_t$, we get an operator

$$\delta = \nabla_{\partial_t} \colon M \to M$$

that still satisfies the Leibniz rule:

$$\delta(fm) = f'm + f\delta(m),$$

where f' is the derivative of f with respect to t. Our goal is to show that M is a free R-module, and since $\dim B = 1$, it suffices to prove that M is torsion-free. So consider the torsion submodule M_{tor}. It is again finitely generated, and because R is local, there is an integer $e \geq 0$ such that $t^e m = 0$ for every $m \in M_{tor}$. Let us choose $e \geq 0$ to be minimal with this property. The Leibniz rule for δ gives

$$0 = \delta(t^e m) = et^{e-1}m + t^e\delta(m),$$

and so $t^{e+1}\delta(m) = -et^e m = 0$. But then $\delta(m) \in M_{tor}$, and so we actually have $t^e\delta(m) = 0$. According to the formula above, $et^{e-1}m = 0$, and the only way this does not contradict the minimality of e is that $e = 0$. But then $M_{tor} = 0$, and so M is indeed a free R-module. ∎

We can now relate $\mathscr{V}^k$ and ∇ to the earlier topological definition. Here the key tool is the base change theorem (whose proof, given Grauert's theorem, is more or less the same as that of the algebraic version in Hartshorne). Recall that following piece of terminology: one says that the sheaf $R^k f_* \Omega^\bullet_{X/B}$ has the *base change property* if the natural morphism

$$\frac{\mathscr{O}_{B,b}}{\mathfrak{m}_b} \otimes_{\mathscr{O}_B} R^k f_* \Omega^\bullet_{X/B} \to H^k\Big(X_b, \Omega^\bullet_{X/B}\big|_{X_b}\Big)$$

is an isomorphism for every $b \in B$. The right-hand side is of course just $H^k(X_b, \mathbb{C})$, because the relative de Rham complex restricted to X_b is isomorphic to $\Omega^\bullet_{X_b}$. So the base change property means that the fiber of the coherent sheaf $\mathscr{V}^k$ at a point $b \in B$ is isomorphic to $H^k(X_b, \mathbb{C})$.

Now each sheaf in the complex $\Omega^\bullet_{X/B}$ is locally free, hence flat over B (because $f: X \to B$ is flat for dimension reasons), and f is proper. In this setting, the base change theorem says the following: If for some integer $k \in \mathbb{Z}$, all the higher direct image sheaves $R^j f_* \Omega^\bullet_{X/B}$ are locally free for $j \geq k+1$, then $R^k f_* \Omega^\bullet_{X/B}$ has the base change property. But we have just seen in Lemma 4.3 that each $\mathscr{V}^k$ is actually locally free, and so they all have the base change property. In particular, $\mathscr{V}^k$ is a holomorphic vector bundle with fibers $H^k(X_b, \mathbb{C})$. Now recall from (4.2) that we have an exact sequence

$$\cdots \to R^k f_* \mathbb{C} \to \mathscr{V}^k \xrightarrow{\nabla} \Omega^1_B \otimes_{\mathscr{O}_B} \mathscr{V}^k \to \cdots$$

Because the fiber of $\mathscr{V}^k$ at a point $b \in B$ is exactly $H^k(X_b, \mathbb{C})$, the morphism $R^k f_* \mathbb{C} \to \mathscr{V}^k$ must be injective, and so $R^k f_* \mathbb{C}$ is isomorphic to the sheaf of ∇-flat sections of $\mathscr{V}^k$, hence locally constant. In particular, ∇ agrees with the Gauss-Manin connection as defined earlier.

Algebraic construction of the Hodge bundles. Now let us construct the Hodge bundles by the same algebraic approach. Suppose for a moment that X is a compact Kähler manifold of dimension n. Then the Hodge-de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_X) \Longrightarrow H^{p+q}(X, \mathbb{C})$$

degenerates at E_1, and the induced filtration is the Hodge filtration on $H^k(X, \mathbb{C})$. The Hodge-de Rham spectral sequence is the spectral sequence of a filtered complex, namely of the de Rham complex $\Omega^\bullet_X$, filtered by the family of subcomplexes

$$0 \to \cdots \to 0 \to \Omega^p_X \to \cdots \to \Omega^n_X \to 0.$$

Let us denote the p-th subcomplex by the symbol $F^p \Omega^\bullet_X$. By general theory, the filtration induced by the spectral sequence is

$$\operatorname{Im}\Big(H^k\big(X, F^p\Omega^\bullet_X\big) \to H^k\big(X, \Omega^\bullet_X\big)\Big),$$

and so the E_1-degeneration means exactly that the morphism

$$H^k\big(X, F^p\Omega^\bullet_X\big) \to H^k\big(X, \Omega^\bullet_X\big)$$

is *injective* for every $k, p \in \mathbb{Z}$. The image therefore agrees with the Hodge filtration on $H^k(X, \mathbb{C})$, and we have

$$H^k\big(X, F^p\Omega^\bullet_X\big) \cong F^p H^k(X, \mathbb{C}). \tag{4.4}$$

Now we return to our family $f: X \to B$. The relative de Rham complex is again filtered by the subcomplexes $F^p \Omega^\bullet_{X/B}$, and if we put everything together, we get a commutative diagram with short exact rows:

$$\begin{tikzcd} 0 \arrow[r] & f^*\Omega_B^1 \otimes_{\mathscr{O}_X} F^{p-1}\Omega_{X/B}^{\bullet-1} \arrow[r] \arrow[d, hook] & F^p\Omega_X^\bullet \arrow[r] \arrow[d, hook] & F^p\Omega_{X/B}^\bullet \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & f^*\Omega_B^1 \otimes_{\mathscr{O}_X} \Omega_{X/B}^{\bullet-1} \arrow[r] & \Omega_X^\bullet \arrow[r] & \Omega_{X/B}^\bullet \arrow[r] & 0 \end{tikzcd}$$

This gives us a commutative diagram for the connecting morphisms:

$$\begin{tikzcd} R^k f_* F^p\Omega_{X/B}^\bullet \arrow[r] \arrow[d] & \Omega_B^1 \otimes_{\mathscr{O}_B} R^k f_* F^{p-1}\Omega_{X/B}^\bullet \arrow[d] \\ R^k f_* \Omega_{X/B}^\bullet \arrow[r] & \Omega_B^1 \otimes_{\mathscr{O}_B} R^k f_* \Omega_{X/B}^\bullet \end{tikzcd} \tag{4.5}$$

The morphism on the bottom is just our connection $\nabla\colon \mathscr{V}^k \to \Omega_B^1 \otimes_{\mathscr{O}_B} \mathscr{V}^k$. The following proposition now explains both why the Hodge bundles are vector bundles, and why they satisfy Griffiths transversality.

Proposition 4.6. *The morphism $R^k f_* F^p\Omega_{X/B}^\bullet \to R^k f_* \Omega_{X/B}^\bullet$ is injective for every $k, p \in \mathbb{Z}$. The image is a holomorphic subbundle $F^p\mathscr{V}^k$, whose fiber at a point $b \in B$ is isomorphic to $F^p H^k(X_b, \mathbb{C})$.*

Of course, Griffiths transversality relation $\nabla(F^p\mathscr{V}^k) \subseteq \Omega_B^1 \otimes_{\mathscr{O}_B} F^{p-1}\mathscr{V}^k$ is now a simple consequence of the commutative diagram in (4.5). The proposition also gives an alternative proof, without any topology, for the fact that the Hodge numbers

$$\dim H^{p,k-p}(X_b) = \operatorname{rk} F^p\mathscr{V}^k - \operatorname{rk} F^{p+1}\mathscr{V}^k$$

are independent of the point $b \in B$.

Proof. We use the base change theorem and the degeneration of the Hodge-de Rham spectral sequence. Fix an integer $p \in \mathbb{Z}$. Since $R^k f_* F^p\Omega_{X/B}^\bullet = 0$ for $k \gg 0$, we can argue by descending induction on k. Suppose we already know that $R^j f_* F^p\Omega_{X/B}^\bullet$ injects into $\mathscr{V}^k = R^k f_* \Omega_{X/B}^\bullet$ for every $j \geq k+1$, hence is locally free (as $\dim B = 1$). The base change theorem tells us that $R^k f_* F^p\Omega_{X/B}^\bullet$ has the base change property, which means that its fiber at any point $b \in B$ is isomorphic to

$$H^k\Big(X_b, F^p\Omega_{X/B}^\bullet\big|_{X_b}\Big) \cong H^k\big(X_b, F^p\Omega_{X_b}^\bullet\big).$$

On fibers, the morphism $R^k f_* F^p\Omega_{X/B}^\bullet \to R^k f_* \Omega_{X/B}^\bullet$ therefore agrees with

$$H^k\big(X_b, F^p\Omega_{X_b}^\bullet\big) \to H^k\big(X_b, \Omega_{X_b}^\bullet\big),$$

which is injective by the E_1-degeneration of the Hodge-de Rham spectral sequence on the compact Kähler manifold X_b. Since both sheaves are coherent, Nakayama's lemma implies that $R^k f_* F^p\Omega_{X/B}^\bullet \to R^k f_* \Omega_{X/B}^\bullet$ is injective, and that the image is a holomorphic subbundle $F^p\mathscr{V}^k \subseteq \mathscr{V}^k$. By the base change property, this subbundle has fibers $F^p H^k(X_b, \mathbb{C})$, as claimed. □

Polarized variations of Hodge structure. Let me end this lecture by giving the definition of a polarized variation of Hodge structure. It is obtained by taking all the features that we found in the geometric setting. I remind you that B is a complex manifold of dimension 1; in higher dimensions, one needs to add the assumption that ∇ is a flat connection.

Definition 4.7. Let $n \in \mathbb{Z}$. A *polarized variation of Hodge structure of weight n* on B consists of the following data:

(a) A holomorphic vector bundle $\mathscr{V}$ with a connection $\nabla \colon \mathscr{V} \to \Omega_B^1 \otimes_{\mathscr{O}_B} \mathscr{V}$.
(b) A decreasing filtration by holomorphic subbundles $F^p\mathscr{V}$, called the *Hodge filtration*, such that $\nabla(F^p\mathscr{V}) \subseteq \Omega_B^1 \otimes_{\mathscr{O}_B} F^{p-1}\mathscr{V}$.
(c) A hermitian pairing $h \colon \mathscr{V} \otimes_{\mathbb{C}} \overline{\mathscr{V}} \to \mathscr{C}_B^\infty$, which is "flat", in the sense that $dh(s', s'') = h(\nabla s', s'') + h(s', \nabla s'')$ for all local sections $s', s'' \in \mathscr{V}$.

The requirement is that, for every point $b \in B$, the fiber V_b, together with the induced pairing $h_b \colon V_b \otimes_{\mathbb{C}} \overline{V_b} \to \mathbb{C}$ and the filtration by the subspaces $F^pV_b \subseteq V_b$, must come from a Hodge structure of weight n.

Here "come from" means that each V_b has a Hodge structure of weight n, say

$$V_b = \bigoplus_{p+q=n} V_b^{p,q},$$

which is polarized by the pairing S_b, such that

$$F^pV_b = \bigoplus_{i \geq p} V_b^{i,n-i}.$$

I will explain next time how the polarization allows one to recover the Hodge decomposition from the Hodge filtration.

Note. The fact that $F^p\mathscr{V}$ is a holomorphic subbundle means that the subquotients

$$\operatorname{gr}_F^p \mathscr{V} = F^p\mathscr{V} / F^{p+1}\mathscr{V}$$

are again locally free. These subquotients are also sometimes referred to as "Hodge bundles".

Exercise 4.2. Suppose that $H = \bigoplus_{p+q=n} H^{p,q}$ is a Hodge structure of weight n, and h a polarization. Explain how one can recover the subspaces $H^{p,q}$ from the Hodge filtration F^pH with the help of h.